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The geometry of concurrence as a measure of entanglement

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Abstract

It is widely recognized that concurrence can be regarded as a measure of two-qubit entanglement. This paper studies the geometry of concurrence for a two-qubit system to show that the concurrence serves as a coordinate of the factor space $G\backslash M \cong [0, 1]$, where $M \cong S^7$ is the space of normalized two-qubit states, and where $G = U(1) \times SU(2) \times SU(2)$. Any monotonically increasing function of the concurrence can serve as a measure for entanglement. From the viewpoint of Riemannian geometry, a state with concurrence r is shown to be distant from the separable states by $\frac{1}{2}\sin^{-1} r$, where r ranges over $0 \leqslant r \leqslant 1$. In addition, measures of entanglement for n-qubits are discussed on the basis of a bipartite decomposition $\mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m}$ with $\ell + m = n$. They are invariant under the local unitary transformation group $U(2^\ell) \times U(2^m)$.

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1. Introduction

Since Bennet *et al* [1] and Hill–Wootter [2] it has been widely recognized that concurrence can serve as a measure of two-qubit entanglement. Geometric study of entanglement has already been made in [3–5], for example. However, the full geometric study of the concurrence for two-qubit systems has not yet been made. It is the purpose of this paper to show that the two-qubit concurrence is characterized completely in terms of transformation groups and geometry; concurrence for a two-qubit system proves to be a coordinate of the factor space $G\backslash M \cong [0, 1]$, where $G = U(1) \times SU(2) \times SU(2)$ and where $M \cong S^7$ is the space of normalized two-qubit states. Further, a state with concurrence r is shown to be distant from the separable states by $\frac{1}{2}\sin^{-1} r$ with $0 \leqslant r \leqslant 1$, with respect to the metric naturally defined on $G\backslash M$ from that on M. In addition, from the viewpoint of transformation groups, candidates for measures of entanglement for n-qubits are put forward on the basis of a bipartite decomposition of the n-qubit system.

The organization of this paper is as follows. Section 2 is a geometric setting for two-qubit states. The state space M for a two-qubit system is identified with the unit sphere S^7, which is realized in the space of 2×2 complex matrices with constraints. The concurrence of a state $C \in M$ is defined to be $|2 \det C|$. The concurrence is clearly invariant under the transformation $C \mapsto e^{i\theta} gCh$ with $e^{i\theta} \in U(1)$ and $g, h \in SU(2)$. This fact and the canonical Riemannian structure defined on S^7 will be used in the succeeding sections. Section 3 contains a review of the Hopf bundle $S^7 \to S^4$. The bundle is realized as $M \to M/SU(2)$. In section 4, the factor space $(SU(2) \times SU(2))\backslash M$ is studied and shown to be homeomorphic with $\overline{D}$, the closed unit disc. The projection $S^7 \cong M \to \overline{D}$ is realized by the map $C \mapsto 2 \det C$. A Riemannian metric on the open subset D of $\overline{D}$ will be obtained by submersing that on M_1, where M_1 is an open dense subset of M. In section 5, the Hopf map $S^7 \to S^4$ is followed by the map $S^4 \to \overline{D}$ to form the projection $S^7 \to \overline{D}$. Section 6 deals with entanglement measurement for two-qubits. It turns out that the factor space $(U(1) \times SU(2) \times SU(2))\backslash M$, on which the concurrence should be defined, is homeomorphic with the closed interval $[0, 1]$. The open interval $(0, 1)$ is endowed with a Riemannian metric, according to which the point $r \in (0, 1)$ is shown to be distant from 0 by $\frac{1}{2} \sin^{-1} r$, where $\sin^{-1}$ denote the arcsine. Put another way, a state C with concurrence $r = |2 \det C|$ is distant from the separable states by $\frac{1}{2} \sin^{-1} r$. Section 7 deals with concurrence as measures of three- and more-qubit entanglement. Section 8 contains concluding remarks and comments.

2. Geometric setting for two-qubit states

The Hilbert space for a two-qubit system is $\mathbf{C}^2 \otimes \mathbf{C}^2$, of which the elements are expressed as $\Psi = \sum c_{jk} \boldsymbol{e}_j \otimes \boldsymbol{e}_k$, where the $\boldsymbol{e}_j$ are the basis vectors of the canonical basis of $\mathbf{C}^2$. The space of the normalized states is characterized by $\sum |c_{jk}|^2 = 1$. Since the matrices $C = (c_{jk})$ with $\operatorname{tr}(C^*C) = \sum |c_{jk}|^2 = 1$ and the normalized states Ψ are in one-to-one correspondence, we take the state space for the two-qubit system as

$$M := \left\{ C = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} \middle| \operatorname{tr}(C^*C) = 1 \right\}, \tag{2.1}$$

which is diffeomorphic with the unit sphere S^7. We note here that the Bell basis for the two-qubit system corresponds to the set of matrices

$$E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$E_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad E_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

It is well known that a normalized state Ψ is maximally entangled if and only if $\sqrt{2}C \in U(2)$, and that Ψ is separable if and only if $\det C = 0$ [5]. In view of this, one of measures for two-qubit entanglement is given by $|\det C|$; Ψ is maximally entangled or separable, according to whether $|\det C| = 1/2$ or $|\det C| = 0$. In [1, 2], the concurrence is defined to be the quantity $|\sum_{i=1}^4 \alpha_i|$, where α_i are determined for $C \in M$ by $C = \sum_{i=1}^4 \alpha_i E_i$ with E_i the matrices given in (2.2). A calculation with this expression of C shows that $2 \det C = \sum_{i=1}^4 \alpha_i^2$, so that one takes $|2 \det C|$ as concurrence.

One of our purposes is to characterize the concurrence in terms of transformation groups. Let U_1 and U_2 be unitary matrices in $U(2)$. Then the local unitary transformation $\Psi \mapsto (U_1 \otimes U_2)\Psi$ gives rise to the action of the group $U(2) \times U(2)$ on M in the manner

$$C \mapsto U_1 C U_2^T, \qquad C \in M, \quad (U_1, U_2) \in U(2) \times U(2). \tag{2.3}$$

In what follows, we work with the action of $G := U(1) \times SU(2) \times SU(2)$ on M, which is described as

$$C \mapsto e^{i\theta} U_1 C U_2^T, \qquad C \in M, \quad e^{i\theta} \in U(1), \quad (U_1, U_2) \in SU(2) \times SU(2). \tag{2.4}$$

The concurrence $|2\det C|$ is manifestly invariant under the action of G. It then turns out that $|2\det C|$ should be defined on the factor space $G\backslash M$. One of our aims is to describe the space $G\backslash M$ in an explicit manner. We have here to refer to [6], in which they state that a measure of entanglement is a function on the space of states of a multiparticle system which is invariant under local unitary operators, i.e., unitary transformations on individual particles. The above group action is already pointed out in [7], but is not associated with the factor space $G\backslash M$.

To measure the entanglement of states, we wish to use a naturally defined metric on $G\backslash M$. To this end, we start with the canonical metric on $M \cong S^7$, which is defined through

$$\langle X_1, X_2\rangle = \tfrac{1}{2}\mathrm{tr}(X_1^* X_2 + X_2^* X_1), \qquad X_1, X_2 \in T_C M, \tag{2.5}$$

where $T_C M$ denotes the tangent space to M at C,

$$T_C M = \{X \in \mathbf{C}^{2\times 2} \mid \mathrm{tr}(C^* X + X^* C) = 0\}, \tag{2.6}$$

and where $\mathbf{C}^{2\times 2}$ denotes the linear space of 2×2 complex matrices. We will find out what metric is defined on the factor space $G\backslash M$ in section 6, which will serve as a measure of entanglement. To be precise, M should be restricted to an open subset of M in order to treat the metric.

3. The Hopf bundle $S^7 \to S^4$ revisited

Before dealing with the space $G\backslash M$, we wish to study the space $M/SU(2)$ or $SU(2)\backslash M$, which will link our study with a preceding work [5] on entanglement measurement associated with the Hopf bundle $S^7 \to S^4$.

The group $SU(2)$ acts on M to the both sides:

$$C \mapsto gC, \quad C \mapsto Cg, \quad g \in SU(2). \tag{3.1}$$

Since these actions are both free, the respective factor spaces, $SU(2)\backslash M$ and $M/SU(2)$, are manifolds. The natural projections are realized as

$$\pi_L : C \mapsto (C^*C, \det C) \in \mathcal{H}_1 \times \mathbf{C}, \qquad \pi_R : C \mapsto (CC^*, \det C) \in \mathcal{H}_1 \times \mathbf{C}, \tag{3.2}$$

respectively, where $\mathcal{H}_1$ denotes the space of 2×2 Hermitian matrices of trace 1. Note here that C^*C and CC^* are invariant under the left and the right $SU(2)$ actions, respectively. We now verify that each factor space is diffeomorphic with S^4. First, we consider the map π_R. Since CC^* is a Hermitian matrix of trace 1, we may put CC^* in the form

$$CC^* = \frac{1}{2}\begin{pmatrix} 1+t & w \\ \overline{w} & 1-t \end{pmatrix}, \qquad w \in \mathbf{C}, \quad t \in \mathbf{R}. \tag{3.3}$$

Further, we set $2\det C = z$. Then, from $\det(CC^*) - |\det C|^2 = 0$, we obtain the equation $1 = t^2 + |w|^2 + |z|^2$, which defines the unit sphere $S^4 \subset \mathbf{R}^5 \cong \mathbf{C}^2 \times \mathbf{R}$. Thus, π_R proves to be a map $S^7 \to S^4$, which is surjective, as is verified by a straightforward calculation. We now show that for a given point $p \in S^4 \subset \mathcal{H}_1 \times \mathbf{C}$, the inverse image $\pi_R^{-1}(p)$ is diffeomorphic with $SU(2)$. Assume that there are matrices $C_1, C_2 \in M$ such that $C_1 C_1^* = C_2 C_2^*$ and $\det C_1 = \det C_2$. Then, from $C_1 C_1^* = C_2 C_2^*$, there exists a unitary matrix g which brings a positive semi-definite matrix $C_1 C_1^* = C_2 C_2^*$ into a diagonal one, $C_1 C_1^* = C_2 C_2^* = g\Lambda^2 g^{-1}$, where Λ^2 is a positive semi-definite diagonal matrix. Then, one has singular decompositions of C_1 and C_2 in the form $C_1 = g\Lambda h_1, C_2 = g\Lambda h_2$, respectively, where Λ is a positive semi-definite diagonal matrix, and $h_1, h_2 \in U(2)$. Hence, we obtain

$$C_2 = g\Lambda h_2 = C_1 h, \qquad h := h_1^{-1} h_2. \tag{3.4}$$

Further, we obtain $\det h = 1$ from $\det C_2 = \det C_1$. This implies that $\pi_R^{-1}(p) \cong SU(2)$. Thus π_R realizes the Hopf bundle $S^7 \to S^4$ with fibre $SU(2)$. In the same manner, we can verify that π_L also provides the Hopf bundle $S^7 \to S^4$. In [3, 5], the Hopf bundle $S^7 \to S^4$ is treated in terms of quaternion. They claim that the Hopf map is entanglement sensitive. However, we would like to say that the map $M \to G\backslash M$ is of more help than the Hopf map $M \to SU(2)\backslash M$ (section 5).

We proceed to the canonical connection on the bundles $\pi_L : S^7 \to S^4$ and $\pi_R : S^7 \to S^4$, respectively. The vertical subspaces of $T_C M$ with respect to the left and right actions are given by

$$V_C^L = \{\xi C | \xi \in su(2)\}, \qquad V_C^R = \{C\xi | \xi \in su(2)\}, \tag{3.5}$$

respectively. We define the horizontal subspaces H_C^L and H_C^R to be the orthogonal complements, $H_C^L = \left(V_C^L\right)^\perp$ and $H_C^R = \left(V_C^R\right)^\perp$, of V_C^L and V_C^R, respectively, with respect to the Riemannian metric given in (2.5). Then, a straightforward calculation shows that H_C^L and H_C^R are given by

$$H_C^L = \{X \in T_C M | CX^* - XC^* \in \mathrm{span}_{\mathbf{R}}\{iI_2\}\}, \tag{3.6}$$

$$H_C^R = \{X \in T_C M | C^* X - X^* C \in \mathrm{span}_{\mathbf{R}}\{iI_2\}\}, \tag{3.7}$$

respectively, where I_2 denotes the 2×2 unit matrix.

To get the horizontal subspace in an explicit manner, we consider the $SU(4)$ action on $M \cong S^7$. Since $SU(4)$ acts transitively on $M \subset \mathbf{C}^2 \otimes \mathbf{C}^2 \cong \mathbf{C}^4$, tangent vectors to M will be obtained by the $su(4)$ action on M. It is well known [8] that $su(4)$ has the Cartan decomposition

$$su(4) = \boldsymbol{k} \oplus \boldsymbol{p}, \tag{3.8}$$

where

$$\boldsymbol{k} = \mathrm{span}\{\mathrm{i}I \otimes \sigma_j/2, \mathrm{i}\sigma_k \otimes I/2\}, \qquad j, k = 1, 2, 3, \tag{3.9}$$

$$\boldsymbol{p} = \mathrm{span}\{\mathrm{i}\sigma_j \otimes \sigma_k/2\}, \qquad j, k = 1, 2, 3, \tag{3.10}$$

and where σ_j are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.11}$$

We note here that $\xi \otimes \eta$ acts on M in the manner that $C \mapsto \xi C \eta^T$, and that $\boldsymbol{k}$ and $\boldsymbol{p}$ satisfy

$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k}, \qquad [\boldsymbol{p}, \boldsymbol{k}] \subset \boldsymbol{p}, \qquad [\boldsymbol{p}, \boldsymbol{p}] \subset \boldsymbol{k}. \tag{3.12}$$

We can verify that the subalgebra $\boldsymbol{k}$ generates the vertical subspace $V_C^L + V_C^R$. In fact, we observe that $\mathrm{i}I \otimes \sigma_j/2$ and $\mathrm{i}\sigma_k \otimes I$ yield vertical tangent vectors

$$\frac{\mathrm{i}}{2} C\sigma_j^T \in V_C^R, \qquad \frac{\mathrm{i}}{2}\sigma_k C \in V_C^L,$$

respectively. In what follows, the singular decomposition $C = g\Lambda h$ is of great help, where $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$ with μ_k singular values of C and where $g, h \in U(2)$. We observe from the singular decomposition of C that the vertical vectors $\xi\Lambda$ and $\Lambda\eta^T$ at $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$, with $\xi, \eta \in su(2)$, are carried to vertical vectors at $C = g\Lambda h$ by $\mathrm{L}_g \circ \mathrm{R}_h$. In fact, one has

$$g(\xi\Lambda)h = \mathrm{Ad}_g(\xi)C, \qquad g(\Lambda\eta^T)h = C\mathrm{Ad}_{h^T}(\eta)^T,$$

where it is to be noted that $\mathrm{Ad}_g : su(2) \to su(2)$ if $g \in U(2)$.

We turn to the horizontal subspace at $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$, and then proceed to the horizontal subspace at $C = g\Lambda h$. While $\boldsymbol{k}$ is associated with vertical vectors, horizontal vectors will be obtained by the action of $\boldsymbol{p}$. We have candidates, $\mathrm{i}\sigma_j\Lambda\sigma_k^T$, $j, k = 1, 2, 3$, for horizontal vectors at Λ. Our task is now to ask if these vectors are horizontal or not. A straightforward calculation results in

$$H_\Lambda^R = \mathrm{span}_{\mathbf{R}}\{X_1, X_2, X_3, X_4\}, \tag{3.13}$$

$$H_\Lambda^L = \mathrm{span}_{\mathbf{R}}\{X_1, X_2, X_5, X_6\}, \tag{3.14}$$

where

$$\begin{aligned} X_1 &= \mathrm{i}\sigma_1\Lambda\sigma_1^T, & X_2 &= \mathrm{i}\sigma_1\Lambda\sigma_2^T, & X_3 &= \mathrm{i}\sigma_1\Lambda\sigma_3^T, \\ X_4 &= \mathrm{i}\sigma_2\Lambda\sigma_3^T, & X_5 &= \mathrm{i}\sigma_3\Lambda\sigma_1^T, & X_6 &= \mathrm{i}\sigma_3\Lambda\sigma_2^T. \end{aligned} \tag{3.15}$$

The horizontal subspace at Λ is carried to that at $C = g\Lambda h$ by $\mathrm{L}_g \circ \mathrm{R}_h$, which can be shown in a straightforward manner:

$$H_C^L = gH_\Lambda^L h, \qquad H_C^R = gH_\Lambda^R h. \tag{3.16}$$

We are now interested in the orthogonality of these horizontal vectors. A straightforward calculation shows that

$$\langle X_k, X_\ell\rangle = \delta_{k\ell}, \qquad k,\ell \in \{1,2,3,4\}, \quad \text{or} \quad k,\ell \in \{1,2,5,6\}, \tag{3.17}$$

where X_k denote the basis vectors in H_Λ^R or in H_Λ^L, according to whether $k \in \{1,2,3,4\}$ or $k \in \{1,2,5,6\}$. Equation (3.17) is true for H_C^R and for H_C^L. In fact, as easily seen, for tangent vectors X and Y at Λ, one has

$$\langle gXh, gYh\rangle_C = \langle X, Y\rangle_\Lambda. \tag{3.18}$$

4. $SU(2) \times SU(2)$ action

We now consider the left and the right $SU(2)$ actions simultaneously,

$$C \mapsto gCh^T \qquad (g,h) \in SU(2) \times SU(2), \tag{4.1}$$

which is a restriction of the map (2.4). Since the group $U(1)$ is easy to treat, we study the above map in this section, and proceed to the full map (2.4) in section 6.

We start by obtaining the isotropy subgroup of $SU(2) \times SU(2)$ at $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$ with $\mu_1 \neq \mu_2, \mu_j \geqslant 0$. Let $(g,h) \in SU(2) \times SU(2)$ and $\Lambda = \mathrm{diag}(\mu_1,\mu_2)$ with $\mu_1 > \mu_2 \geqslant 0$. Then, the equation $g_0\Lambda h_0^T = \Lambda$ is shown to be solved by

$$g_0 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\chi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\chi} \end{pmatrix}, \qquad h_0 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\chi} \end{pmatrix} = g_0^{-1}. \tag{4.2}$$

Hence the isotropy subgroup at $\Lambda = \mathrm{diag}(\mu_1,\mu_2)$ with $\mu_1 > \mu_2 \geqslant 0$ proves to be

$$G_\Lambda = \left\{\left(g_0, g_0^{-1}\right)\middle| g_0 = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\chi}, \mathrm{e}^{-\mathrm{i}\chi})\right\} \cong U(1). \tag{4.3}$$

If $\mu := \mu_1 = \mu_2 > 0$, the isotropy subgroup at $\Lambda = \mu I$ is given by

$$G_\Lambda = \{(g, \overline{g}) | g \in SU(2)\} \cong SU(2). \tag{4.4}$$

For a generic $C \in M$, we put C in the form $C = g\Lambda h$, $\Lambda = \mathrm{diag}(\mu_1,\mu_2)$ with $g, h \in U(2)$, where $\mu_k \geqslant 0$ are the singular values of C. Then, for $(g_0, h_0) \in G_\Lambda$ with $\mu_1 \neq \mu_2$, one obtains $\mathrm{A}_g(g_0)C(\mathrm{A}_{h^T}(h_0))^T = C$, where A_g denotes the inner automorphism, $\mathrm{A}_g(k) = gkg^{-1}$. This implies that

$$G_C = \{(\mathrm{A}_g(g_0), \mathrm{A}_{h^T}(h_0)) | (g_0, h_0) \in G_\Lambda\} \cong U(1). \tag{4.5}$$

The same reasoning is true if $\mu_1 = \mu_2$, and thereby resulting in $G_C \cong SU(2)$. It turns out that the isotropy subgroup of $SU(2) \times SU(2)$ at C is $U(1)$ or $SU(2)$ according to whether $\mu_1 \neq \mu_2$ or $\mu_1 = \mu_2$:

$$G_C \cong \begin{cases} U(1) & \text{if} \quad \mu_1(C) \neq \mu_2(C), \\ SU(2) & \text{if} \quad \mu_1(C) = \mu_2(C), \end{cases} \tag{4.6}$$

where $\mu_k(C), k = 1, 2$, denote the singular values of $C \in M$.

According to whether the singular values are different or not, the state space $M \cong S^7$ is broken up into two subsets,

$$M = M_1 \cup M_2, \tag{4.7}$$

where

$$M_1 = \{C \in M | \mu_1(C) \neq \mu_2(C)\}, \qquad M_2 = \{C \in M | \mu_1(C) = \mu_2(C)\}. \tag{4.8}$$

These subsets are invariant under the $SU(2) \times SU(2)$ action, since the singular values are invariant under the transformation $C \mapsto gCh^T$ with $(g, h) \in SU(2) \times SU(2)$.

We are to look into the invariant subsets, M_1 and M_2. First we take up M_2. Then, $C \in M_2$ is expressed as $C = g\Lambda h$ with $\Lambda = \text{diag}(\mu_1, \mu_2), \mu_1 = \mu_2$, and $g, h \in U(2)$. Since $\text{tr}(C^*C) = \mu_1^2 + \mu_2^2 = 1$, one has $\mu_1^2 = \mu_2^2 = 1/2$, so that $\sqrt{2}C = gh \in U(2)$. This implies that $M_2 \cong U(2)$. The isotropy subgroup at $C \in M_2$ is already known to be a subgroup isomorphic with $SU(2)$. Hence, the orbit through $C \in M_2$ is diffeomorphic with $(SU(2) \times SU(2))/SU(2) \cong SU(2)$, so that the orbit space proves to be

$$SU(2)\backslash M_2 \cong SU(2)\backslash U(2) \cong U(1). \tag{4.9}$$

We turn to M_1. Since the isotropy subgroup at $C \in M_1$ is isomorphic with $U(1)$, the orbit through C is diffeomorphic with $(SU(2) \times SU(2))/U(1)$. We wish to know of the orbit space $(SU(2) \times SU(2))\backslash M_1$. To this end, we take up $\det(C)$, which is invariant under the $SU(2) \times SU(2)$ action. On account of the constraint that $\text{tr}(C^*C) = 1$, one has

$$\det(C^*C) = \mu_1^2\mu_2^2 \leqslant \left(\frac{\mu_1^2 + \mu_2^2}{2}\right)^2 = \left(\frac{1}{2}\text{tr}(C^*C)\right)^2 = \frac{1}{4}, \tag{4.10}$$

so that

$$|\det(C)| \leqslant \tfrac{1}{2}. \tag{4.11}$$

The equality occurs if and only if $\mu_1 = \mu_2$. Put another way, $|\det(C)| = 1/2$ if and only if $C \in M_2$. Hence, for $C \in M_1$, we have $|\det(C)| < 1/2$.

We are allowed to regard $2\det(C) = z$ as the map

$$2\det : M_1 \longrightarrow D := \{z \in \mathbf{C} | |z| < 1\}. \tag{4.12}$$

We show that this map is surjective. For a given $z = r\mathrm{e}^{\mathrm{i}\theta} \in D$ with $0 \leqslant r < 1$, we have to solve the equation $2\det(C) = z$. To this end, we choose to look for C in the form of singular decomposition, $C = g\Lambda h$, where $g, h \in U(2)$ and $\Lambda = \text{diag}(\mu_1, \mu_2)$. Let g_0, h_0, and Λ_0 be unitary matrices and a diagonal matrix, respectively, such that $\det(g_0) = \mathrm{e}^{\mathrm{i}(\theta-\alpha)/2}, \det(h_0) = \mathrm{e}^{\mathrm{i}(\theta+\alpha)/2}$, and $\det(\Lambda_0) = r/2$, where α is an undetermined real number. Then, the matrix $C_0 = g_0\Lambda_0 h_0$ gives a solution to $2\det(C) = z$. This means that the map $2\det : M_1 \to D$ is surjective. We note here that Λ_0 is unique for a given z, if we choose μ_1 to be greater than μ_2 ($\mu_1 > \mu_2 \geqslant 0$). This is because the singular values μ_1, μ_2, which are subject to $\mu_1^2 + \mu_2^2 = 1, \mu_1^2\mu_2^2 = r^2/4$, are distinct on account of $r < 1$. In particular, if $r = 0$ then $\Lambda_0 = \text{diag}(1, 0)$. We proceed to explore the inverse image $\det^{-1}(z)$ of $z \in D$. Suppose that for a given $z = r\mathrm{e}^{\mathrm{i}\theta} \in D$, there are two solutions $C_1 = g_1\Lambda h_1$ and

$C_2 = g_2 \Lambda h_2$ such that $\det(C_1) = \det(C_2) = z/2$. Then, one has $\det(g_1 h_1) = \det(g_2 h_2) = \mathrm{e}^{\mathrm{i}\theta}$, if $\det\Lambda \neq 0$. From this, it follows that $\det\left(g_2^{-1} g_1\right) = \det\left(h_2 h_1^{-1}\right) = \mathrm{e}^{\mathrm{i}\alpha}$, where α is a real number, and where we have used the fact that $g_k, h_k \in U(2), k = 1, 2$. Hence, we obtain $\det\left(\mathrm{e}^{-\mathrm{i}\alpha/2} g_2^{-1} g_1\right) = \det\left(\mathrm{e}^{-\mathrm{i}\alpha/2} h_2 h_1^{-1}\right) = 1$. This implies that there are $g, h \in SU(2)$ such that $\mathrm{e}^{-\mathrm{i}\alpha/2} g_2^{-1} g_1 = g^{-1}$ and $\mathrm{e}^{-\mathrm{i}\alpha/2} h_2 h_1^{-1} = h$. Thus, one has

$$g_2 = \mathrm{e}^{-\mathrm{i}\alpha/2} g_1 g, \qquad h_2 = \mathrm{e}^{\mathrm{i}\alpha/2} h h_1, \qquad g, h \in SU(2). \tag{4.13}$$

Hence, two solutions C_1 and C_2 are related by

$$C_2 = g_2 \Lambda h_2 = g_1 g g_1^{-1} C_1 h_1^{-1} h h_1, \tag{4.14}$$

where $g_1 g g_1^{-1}, h_1^{-1} h h_1 \in SU(2)$, though $g_k, h_k \in U(2)$. This equation implies that two solutions, C_1 and C_2, are related by the $SU(2) \times SU(2)$ action. We need to look into this action in detail. As was already proved in (4.5), the $SU(2) \times SU(2)$ action has the isotropy subgroup isomorphic with $U(1)$. This implies that two solutions, C_1 and C_2, are related by the $SU(2) \times SU(2)$ action up to the $U(1)$ action. Put another way, the set of solutions to $2\det(C) = z$ with $z \neq 0$ is diffeomorphic to $(SU(2) \times SU(2))/U(1)$, the orbit of $SU(2) \times SU(2)$ through a solution C. We now turn to the case of $\det(\Lambda) = 0$. Clearly, for $\Lambda_0 = \mathrm{diag}(1, 0) \in M_1$, one has $\det(\Lambda_0) = 0$. Suppose that there is another solution $C \in M_1$ such that $\det(C) = 0$. Then, C is decomposed into $C = g\Lambda_0 h$ with $g, h \in U(2)$, which means that two solutions, Λ_0 and C, are related by the $U(2) \times U(2)$ action. For $g, h \in U(2)$, we may take $g' = g\mathrm{e}^{\mathrm{i}\theta}$ and $h' = h\mathrm{e}^{\mathrm{i}\phi}$ as matrices in $SU(2)$. Then, we obtain

$$C = g' \,\mathrm{e}^{-\mathrm{i}\theta} \Lambda_0 \,\mathrm{e}^{-\mathrm{i}\phi} h' = g' \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} h'. \tag{4.15}$$

Since $g'\mathrm{diag}(\mathrm{e}^{-\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta})$ and $\mathrm{diag}(\mathrm{e}^{-\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\phi})h'$ are both in $SU(2)$, the above equation shows that C and Λ_0 are related by the $SU(2) \times SU(2)$ action. This action is not free. In fact, we can prove that $g_3 \Lambda_0 h_3 = \Lambda_0$ with $g_3, h_3 \in SU(2)$ if and only if $g_3 = h_3^{-1} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\chi}, \mathrm{e}^{-\mathrm{i}\chi})$. This implies that $\det^{-1}(0) \cong (SU(2) \times SU(2))/U(1)$. It then turns out that, for any z with $|z| < 1$, $\det^{-1}(z/2)$ is diffeomorphic with the orbit of $C \in M_1$ along with $2\det(C) = z$. Thus we have shown that the orbit space for M_1 is diffeomorphic with D,

$$(SU(2) \times SU(2))\backslash M_1 \cong D. \tag{4.16}$$

We have already known that the orbit space for M_2 is diffeomorphic with $U(1)$ and that $\det(C) = 1$ if and only if $C \in M_2$. This means that the orbit space $U(1) \cong S^1$ is realized as the boundary of D. Thus, we have proved the following:

Theorem 1. *The orbit space for the whole state space $M \cong S^7$ is homeomorphic with the closed disc:*

$$(SU(2) \times SU(2))\backslash M \cong \overline{D}. \tag{4.17}$$

For the purpose of entanglement measurement, we study the metric on $D \cong (SU(2) \times SU(2))\backslash M_1$ which comes from that on $M_1 \subset M \cong S^7$. The tangent space to the orbit of $SU(2) \times SU(2)$ at $C \in M$ is spanned by

$$\xi C + C\eta^T, \qquad (\xi, \eta) \in su(2) \times su(2), \tag{4.18}$$

and described as $V_C^L + V_C^R$ (see (3.5) for the definition of V_C^L and V_C^R). For $C \in M_1$, one verifies that

$$\begin{aligned} V_C^L \cap V_C^R &= \{\xi C = C\eta^T \mid (\xi, \eta) \in su(2) \times su(2)\} \\ &\cong \{(\xi_0, -\xi_0) \mid \xi_0 = \mathrm{diag}(\chi, -\chi), \chi \in \mathbf{R}\} \\ &\cong \mathcal{G}_C \cong u(1), \end{aligned} \tag{4.19}$$

where $\mathcal{G}_C$ denotes the Lie algebra of the isotropy subgroup G_C given in (4.6) with $\mu_1(C) \neq \mu_2(C)$. Hence, one has $\dim\left(V_C^L + V_C^R\right) = 6 - 1 = 5$, the dimension of the orbit $(SU(2) \times SU(2))/U(1)$ through $C \in M_1$. The horizontal subspace at $C \in M_1$ is given by $\left(V_C^L + V_C^R\right)^\perp = H_C^L \cap H_C^R$, of which the dimension is $\dim\left(V_C^L + V_C^R\right)^\perp = 7 - 5 = 2 = \dim\left(H_C^L \cap H_C^R\right)$. From (3.13) and (3.14), it follows that

$$H_C^L \cap H_C^R = g\left(H_\Lambda^L \cap H_\Lambda^R\right)h = \{gX_1h, gX_2h\}, \tag{4.20}$$

where X_1 and X_2 are given by (3.15). As was shown in (3.17) and (3.18), the horizontal vectors gX_1h, gX_2h form an orthonormal system.

The factor space $D \cong (SU(2)\times SU(2))\backslash M_1$ is endowed with a Riemannian metric through the map $2\det : M_1 \to (SU(2) \times SU(2))\backslash M_1$ so that it may be a Riemannian submersion. Put another way, the Riemannian metric $d\sigma^2$ on D is defined through

$$\langle (2\det_*)_C X, (2\det_*)_C Y\rangle_{2\det(C)} = \langle X, Y\rangle_C, \tag{4.21}$$

where $X, Y \in H_C^L \cap H_C^R$. To find the explicit expression of $\mathrm{d}\sigma^2$, we have to know the expression of the tangent map $\det_*$. However, it is easy to find that

$$(\det_*)_C X = \det(C)\mathrm{tr}(C^{-1}X), \qquad X \in T_C M. \tag{4.22}$$

We now verify that the horizontal subspace $H_C^L \cap H_C^R$ at $C \in M_1$ maps isomorphically to the tangent space to D at $z = 2\det(C)$, if $0 < |z| < 1$. A straightforward calculation along with (4.20) provides

$$U_1 := (2\det_*)_C(gX_1h) = \frac{2\mathrm{i}z}{|z|}, \tag{4.23}$$

$$U_2 := (2\det_*)_C(gX_2h) = -\frac{2z\sqrt{1-|z|^2}}{|z|}, \tag{4.24}$$

which shows that $(2\det_*)_C$ is a vector space isomorphism of the horizontal subspace at C with the tangent space to D at $z = 2\det C$ with $0 < |z| < 1$. Further, from definition (4.21), these vector fields should be orthonormal to each other with respect to the metric $\mathrm{d}\sigma^2$ on D, $\mathrm{d}\sigma^2(U_j, U_k) = \delta_{jk}$, $j, k = 1, 2$. From (4.23) and (4.24), the vectors U_k, $k = 1, 2$, which are a moving frame on D, proves to be expressed, in terms of the polar coordinates, $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, on D, as

$$U_1 = \frac{2}{r}\frac{\partial}{\partial\theta}, \qquad U_2 = -2\sqrt{1-r^2}\frac{\partial}{\partial r}. \tag{4.25}$$

The metric $\mathrm{d}\sigma^2$ satisfying $\mathrm{d}\sigma^2(U_j, U_k) = \delta_{jk}$ are then given by

$$\mathrm{d}\sigma^2 = \frac{1}{4}\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\theta^2\right). \tag{4.26}$$

Theorem 2. *The open disc D, which is realized as the orbit space $(SU(2) \times SU(2))\backslash M_1$, is endowed with the Riemannian metric given in (4.26).*

5. The map $S^4 \to \overline{D}$

So far we have studied the maps $S^7 \to S^4$ and $S^7 \to \overline{D}$. We are now interested in the map $S^4 \to \overline{D}$. Recall that S^4 is realized by $(CC^*, \det C)$ as in (3.2) and that the variables w and t are defined through (3.3) and the variable z by $z = 2\det C$. Since CC^* and $\det C$ are invariant under the right $SU(2)$ action, the variables $w, z \in \mathbf{C}$ and $t \in \mathbf{R}$ are also invariant under the right $SU(2)$ action, and therefore the quotient space S^4 is described in terms of these invariants.

Though $z = 2\det C$ is invariant under the left $SU(2)$ action as well, w and t are not. We now wish to study the left $SU(2)$ action on the variables w and t. The left $SU(2)$ action on M induces the adjoint action on CC^*; $CC^* \longmapsto gCC^*g^{-1}$, which gives rise to an action on $(w, t) \in \mathbf{C} \times \mathbf{R} \cong \mathbf{R}^3$. First we take $g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, a one-parameter subgroup of $SU(2)$. A straightforward calculation provides

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 1+t & w \\ \overline{w} & 1-t \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} 1+t & e^{2i\theta} w \\ e^{-2i\theta}\overline{w} & 1-t \end{pmatrix}, \tag{5.1}$$

which defines the map

$$w \mapsto e^{2i\theta} w, \qquad t \mapsto t. \tag{5.2}$$

On setting $w = u + iv$, this transformation is expressed as a rotation about the t-axis,

$$\begin{pmatrix} u \\ v \\ t \end{pmatrix} \longmapsto \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix}. \tag{5.3}$$

In a similar manner, with $g = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$ and $g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, we can associate a rotation about the u-axis,

$$\begin{pmatrix} u \\ v \\ t \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix}, \tag{5.4}$$

and a rotation about the v-axis,

$$\begin{pmatrix} u \\ v \\ t \end{pmatrix} \longmapsto \begin{pmatrix} \cos 2\theta & 0 & -\sin 2\theta \\ 0 & 1 & 0 \\ \sin 2\theta & 0 & \cos 2\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ t \end{pmatrix}, \tag{5.5}$$

respectively.

Put together, the rotations (5.3), (5.4), and (5.5) generate any rotation in the space $\mathbf{C} \times \mathbf{R} \cong \mathbf{R}^3$, the (u, v, t)-space. Thus, we have shown that the left $SU(2)$ action on M gives rise to the rotation group $SO(3)$ acting on the (u, v, t)-space.

Since S^4 is given by $|z|^2 + |w|^2 + t^2 = 1$ in $\mathbf{C}^2 \times \mathbf{R}$, and since the induced $SO(3)$ action leaves both z and $|w|^2 + t^2 = u^2 + v^2 + t^2$ invariant, the $SO(3)$ acts indeed on S^4. We now look into the $SO(3)$ action on S^4. If $|z| \neq 1$, then one has a two-sphere $S^2(\sqrt{1-|z|^2})$ of radius $\sqrt{1-|z|^2}$ in S^4 for each fixed $z \in D$. Hence, the punctured sphere $S^4 - \{|z| = 1\}$ is decomposed into

$$S^4 - \{|z| = 1\} = \bigsqcup_{z \in D} \{z\} \times S^2(\sqrt{1-|z|^2}) \cong D \times S^2. \tag{5.6}$$

Since the $SO(3)$ acts transitively on each factor space $S^2(\sqrt{1-|z|^2})$ and leaves D invariant, we obtain the quotient space

$$SO(3)\backslash(S^4 - \{|z| = 1\}) \cong D. \tag{5.7}$$

If we set $|z| = 1$ in S^4, we have a circle $|z| = 1$ with $(w, t) = 0$. The $SO(3)$ leaves invariant z and $(w, t) = 0$, so that one has

$$SO(3)\backslash(S^4 \cap \{|z| = 1\}) \cong \{z \in \mathbf{C} \,|\, |z| = 1\} \cong S^1. \tag{5.8}$$

Equations (5.7) and (5.8) are put together to show that the total quotient space is homeomorphic to $\overline{D}$,

$$SO(3)\backslash S^4 \cong \overline{D} = \{z \in \mathbf{C} \,|\, |z| \leqslant 1\}. \tag{5.9}$$

Thus, we have the following:

Theorem 3. *The Hopf bundle $S^7 \to S^4$ is followed by the map $S^4 \to \overline{D}$ to accomplish the following diagram,*

$$\begin{array}{ccc} S^7 & \rightarrow & S^4 \\ \downarrow & \swarrow & \\ \overline{D} & & \end{array}, \tag{5.10}$$

where the maps indicated by the down-arrow and by the right-arrow have been studied in sections 4 and 3 (in the name of π_R), respectively, and where the map assigned by the SW-arrow denotes the projection, $S^4 \to SO(3)\backslash S^4 \cong \overline{D}$, given in (5.9).

In conclusion of this section, we study the metric on S^4 which is defined from that on M, and further investigate how the metrics on S^4 and on D are related to each other. We start with the eigenvalues of the matrix CC^*. From $|CC^* - \lambda I_2| = 0$, we find that they are given by

$$\lambda_1 = \tfrac{1}{2}\left(1 + \sqrt{t^2 + |w|^2}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(1 - \sqrt{t^2 + |w|^2}\right). \tag{5.11}$$

Hence, CC^* is put in the form

$$CC^* = \frac{1}{2}\begin{pmatrix} 1+t & w \\ \overline{w} & 1-t \end{pmatrix} = g\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} g^*, \tag{5.12}$$

where $g \in U(2)$. From this, it follows that $g = I_2$ if and only if $w = 0$ and $t > 0$. We take S^4 as realized by $|z|^2 + |w|^2 + t^2 = 1$ in $\mathbf{C}^2 \times \mathbf{R}$ or by $x^2 + y^2 + u^2 + v^2 + t^2 = 1$ in $\mathbf{R}^5$ with $z = x + iy$.

Let ρ denote the map $C \mapsto CC^*$. Then, the differential of ρ is given by

$$\rho_*(X) = XC^* + CX^*, \qquad X \in T_C M. \tag{5.13}$$

The differential of the map $\pi_R : M \to S^4$ is then given by $\pi_{R*} = (\rho_*, \det_*)$, where $\det_*$ is already given in (4.22). Like (4.21), a metric on S^4 is defined through

$$\langle (\pi_{R*})_C X, (\pi_{R*})_C Y \rangle_{\pi_R(C)} = \langle X, Y \rangle_C, \qquad X, Y \in H_C^R. \tag{5.14}$$

We are to carry horizontal vectors in H_C^R to tangent vectors to S^4 by π_{R*}. Recall that the horizontal subspace H_Λ^R at $\Lambda = \mathrm{diag}(\mu_1, \mu_2)$ is given by (3.13) and H_C^R at $C = g\Lambda h$ by (3.16). There are four linearly independent vectors $gX_k h$ in H_C^R, for which we are going to calculate $\pi_{R*}(gX_k h)$, $k = 1, \ldots, 4$. We have already found out $(\det_*)_C(gX_1 h)$ and $(\det_*)_C(gX_2 h)$ in (4.23) and (4.24), respectively. Further, it is easy to verify that

$$(\det_*)_C(gX_3 h) = (\det_*)_C(gX_4 h) = 0. \tag{5.15}$$

The remaining task to do is to calculate $(\rho_*)_C(gX_k h)$, $k = 1, \ldots, 4$. It is a matter of straightforward calculation to obtain

$$(\rho_*)_C(gX_1 h) = 0, \tag{5.16a}$$

$$(\rho_*)_C(gX_2 h) = |z| g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^*, \tag{5.16b}$$

$$(\rho_*)_C(gX_3h) = g\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g^*, \tag{5.16c}$$

$$(\rho_*)_C(gX_4h) = -g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^*. \tag{5.16d}$$

In view of (5.12), the right-hand sides of (5.16b), (5.16c), and (5.16d) are regarded as tangent vectors to $\mathcal{H}_1$ at $\frac{1}{2}\mathrm{Ad}_g\begin{pmatrix} 1+|t| & 0 \\ 0 & 1-|t| \end{pmatrix}$ in the directions of $(\mathrm{Ad}_g)_*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $(\mathrm{Ad}_g)_*\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, and$(\mathrm{Ad}_g)_*\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, respectively, where$(\mathrm{Ad}_g)_*$ denotes the differential of Ad_g. In terms of the local coordinates (u, v, t) of $\mathcal{H}_1 \cong \mathbf{R}^3$, these tangent vectors are then expressed as

$$(\rho_*)_C(gX_2h) = 2|z|(\mathrm{Ad}_g)_*\left(\frac{\partial}{\partial t}\right)_q, \tag{5.17a}$$

$$(\rho_*)_C(gX_3h) = -2(\mathrm{Ad}_g)_*\left(\frac{\partial}{\partial v}\right)_q, \tag{5.17b}$$

$$(\rho_*)_C(gX_4h) = -2(\mathrm{Ad}_g)_*\left(\frac{\partial}{\partial u}\right)_q, \tag{5.17c}$$

respectively, where $q = (0, 0, t)$ with $t > 0$. It should be noted that Ad_g defines the $SO(3)$ action on $\mathcal{H}_1 \cong \mathbf{R}^3$.

We here recall that if $|z| \neq 1$, then $S^4 - \{|z| = 1\}$ is decomposed as in (5.6). In view of this decomposition, we are to treat the metric induced on the sphere $S^2(\sqrt{1-|z|^2})$. When restricted to $S^2(\sqrt{1-|z|^2})$, definition (5.14) provides

$$\langle (\rho_*)_C(gX_kh), (\rho_*)_C(gX_\ell h)\rangle = \langle gX_kh, gX_\ell h\rangle_C, \qquad k, \ell = 3, 4, \tag{5.18}$$

where the brackets in the left-hand side denote the metric on $S^2(\sqrt{1-|z|^2})$. We have here to note that $(\rho_*)_C(gX_2h)$ is normal to the sphere $S^2(\sqrt{1-|z|^2})$, so that it makes no contribution to determining the metric on $S^2(\sqrt{1-|z|^2})$. Since $\langle gX_kh, gX_\ell h\rangle = \langle X_k, X_\ell\rangle$, as is easily seen, equations (5.17b), (5.17c) and (5.18) are put together to provide

$$\left\langle -2(\mathrm{Ad}_g)_*\left(\frac{\partial}{\partial v}\right)_q, -2(\mathrm{Ad}_g)_*\left(\frac{\partial}{\partial u}\right)_q\right\rangle = \left\langle -2\left(\frac{\partial}{\partial v}\right)_q, -2\left(\frac{\partial}{\partial u}\right)_q\right\rangle, \text{etc}, \tag{5.19}$$

where $q = (0, 0, t)$ with $t = \sqrt{1-|z|^2}$. This implies that the metric on $S^2(\sqrt{1-|z|^2})$ should be $SO(3)$ invariant and determined by the inner product on the tangent space at q. Since $\langle gX_kh, gX_\ell h\rangle = \delta_{k\ell}$, we obtain

$$\left\langle\left(\frac{\partial}{\partial v}\right)_q, \left(\frac{\partial}{\partial u}\right)_q\right\rangle = 0, \qquad \left\langle\left(\frac{\partial}{\partial u}\right)_q, \left(\frac{\partial}{\partial u}\right)_q\right\rangle = \left\langle\left(\frac{\partial}{\partial v}\right)_q, \left(\frac{\partial}{\partial v}\right)_q\right\rangle = \frac{1}{4}. \tag{5.20}$$

Since the metric defined on the sphere $S^2(\sqrt{1-|z|^2})$ is $SO(3)$ invariant, it turns out to be given by

$$\tfrac{1}{4}(1-|z|^2)\,\mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 := \mathrm{d}\phi^2 + \sin^2\phi\,\mathrm{d}\psi^2, \tag{5.21}$$

where $\mathrm{d}\Omega^2$ denotes the canonical metric on the unit sphere S^2. The above metric is also induced on $S^2(\sqrt{1-|z|^2})$ from the metric $\frac{1}{4}(\mathrm{d}u^2+\mathrm{d}v^2+\mathrm{d}t^2)$ by setting $u+\mathrm{i}v = R\,\mathrm{e}^{i\psi}\sin\phi$, $t = R\cos\phi$ with $R = \sqrt{1-|z|^2}$.

So far we have obtained the metric on the factor space S^2 of $S^4 - \{|z| = 1\} \cong D \times S^2$. The metric defined on the factor space D has been already obtained in (4.26). Since the systems

$\{gX_1h, gX_2h\}$ and $\{gX_3, h, gX_4h\}$ are orthogonal to each other and since $\{gX_1h, gX_2h\}$ and $\{gX_3h, gX_4h\}$ determine the metrics on D and on S^2, respectively, these metrics are put together to provide the metric on $S^4 - \{|z| = 1\}$,

$$ds^2 = \frac{1}{4}\left(\frac{dr^2}{1-r^2} + r^2 d\theta^2\right) + \frac{1}{4}(1-r^2)\, d\Omega^2. \tag{5.22}$$

We note here that this metric is induced on S^4 from the flat metric $\frac{1}{4}(dx^2+dy^2+du^2+dv^2+dt^2)$ on $\mathbf{R}^5$ through

$$x + iy = r\,e^{i\theta}, \qquad u + iv = R\,e^{i\psi}\sin\phi, \qquad t = R\cos\phi, \qquad r^2 + R^2 = 1. \tag{5.23}$$

Thus, the metric given in (5.22) extends to the whole sphere S^4, as is well known.

It was pointed out in [5] that part of (5.22),

$$\frac{1}{4}\left(\frac{dr^2}{1-r^2} + (1-r^2)\, d\Omega^2\right), \tag{5.24}$$

defines the Bures metric on the space of density matrices, i.e., the space of CC^* with $C \in M$.

6. Entanglement measurement for two-qubit

We are now in a position to describe the factor space $G\backslash M$ with $G = U(1) \times SU(2) \times SU(2)$. Since $U(1)$ acts on $\overline{D}$ in the manner, $z \mapsto e^{i\theta}z$, we obtain

$$G\backslash M \cong U(1)\backslash\overline{D} \cong [0, 1], \tag{6.1}$$

where the right-hand side denotes the closed interval. As we anticipated in section 2, the concurrence is defined on $G\backslash M$ and serves also as a coordinate of the closed interval, $r = |z| = |2\det C|$. Since the end points $r = 0$ and $r = 1$ are associated with the separable states and the maximally entangled states, respectively, we are allowed to take any monotonically increasing function of r as a measure of entanglement. A natural measure is defined through a natural metric on $G\backslash M$. The open interval $(0, 1)$ is endowed with the metric determined by that on D. In fact, from (4.26), one obtains

$$d\tau^2 = \frac{1}{4}\frac{dr^2}{1-r^2}. \tag{6.2}$$

The length of the interval $r_1 \leqslant r \leqslant r_2$ with respect to $d\tau^2$ is then given by

$$\int_{r_1}^{r_2} d\tau = \frac{1}{2}\int_{r_1}^{r_2} \frac{dr}{\sqrt{1-r^2}} = \frac{1}{2}(\sin^{-1} r_2 - \sin^{-1} r_1), \tag{6.3}$$

where $\sin^{-1}$ denotes the arcsine with the range $[-\pi/2, \pi/2]$. Letting $r_1 \to 0$, we observe that $r = |2\det C|$ is distant from 0 by $\frac{1}{2}\sin^{-1} r$. Summing up the above, we obtain the following:

Theorem 4. *The orbit space $G\backslash M$ with $G = U(1) \times SU(2) \times SU(2)$ is homeomorphic with the closed interval* $[0, 1]$. *The open subset* $(0, 1)$ *is endowed with the Riemannian metric given by (6.2), with respect to which r is distant from 0 by $\frac{1}{2}\sin^{-1} r$, which means that a two-qubit system C with concurrence $r = |2\det C|$ is distant from the separable states by $\frac{1}{2}\sin^{-1} r$.*

One of well-known measures of entanglement is the von Neumann entropy, which is defined to be

$$S(C) = -\mathrm{tr}(CC^*\log(CC^*)), \tag{6.4}$$

and written also as

$$S(C) = -\sum_k \lambda_k \log \lambda_k, \tag{6.5}$$

where λ_k are the eigenvalues of CC^*. Since the $S(C)$ is invariant under the $U(1) \times SU(2) \times SU(2)$ action, it projects to a function on the closed interval [0, 1] (see (6.1)). In fact, from (6.5) together with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = r^2/4$, one obtains a monotonically increasing function on [0, 1],

$$\widetilde{S}(r) = -\tfrac{1}{2}(1+\sqrt{1-r^2})\log\tfrac{1}{2}(1+\sqrt{1-r^2}) - \tfrac{1}{2}(1-\sqrt{1-r^2})\log\tfrac{1}{2}(1-\sqrt{1-r^2}). \quad (6.6)$$

7. Three- and more-qubit concurrence

In this section, we make some comments on further study of entanglement measurement. The entanglement for three- and more-qubit systems has been studied in many ways [7, 9–17]. Let

$$\Psi = \sum_{j,k,\ell\in\{0,1\}} c_{jk\ell}\boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_\ell \quad (7.1)$$

be a three-qubit state, where $\sum_{j,k,\ell} |c_{jk\ell}|^2 = 1$. Let A be a binary integer variable ranging over {00, 01, 10, 11}. Then the above state is rewritten as

$$\Psi = \sum_{j,A} c_{jA}\boldsymbol{e}_j \otimes \boldsymbol{e}_A, \quad (7.2)$$

where $\boldsymbol{e}_A$ denotes $\boldsymbol{e}_k \otimes \boldsymbol{e}_\ell$. Put another way, the three-qubit Hibert space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ is identified with $\mathbf{C}^2 \otimes \mathbf{C}^4$. We denote the coefficient matrix of (7.2) by

$$C = (c_{jA}) = \begin{pmatrix} c_{000} & c_{001} & c_{010} & c_{011} \\ c_{100} & c_{101} & c_{110} & c_{111} \end{pmatrix} \in \mathbf{C}^{2\times 4}, \quad (7.3)$$

where $\mathbf{C}^{2\times 4}$ is the linear space of 2×4 complex matrices. Since the state Ψ is normalized, the C is subject to the constraint $\mathrm{tr}(CC^*) = 1$. Now, the state Ψ is separable in the sense that Ψ is a tensor product of the first one-qubit state and the last two-qubit state, if and only if C is of rank 1. On account of the constraint $\mathrm{tr}(CC^*) = 1$, the rank of CC^* is 1 or 2, so that CC^* is of rank 1 if and only if $\det(CC^*) = 0$. Since C and CC^* has the same rank, we may take $\det(CC^*)$ as a measure of entanglement. We note here that if $C \in \mathbf{C}^{2\times 2}$ this quantity reduces to $|\det C|^2$, the square of the two-qubit concurrence up to a constant factor.

We now show that the $\det(CC^*)$ is invariant under the $U(2) \times U(4)$ action, where $U(2) \times U(4)$ acts on the state space by

$$(U \otimes V)\Psi = \sum_{i,j,A,B} u_{ji}c_{iA}v_{BA}\boldsymbol{e}_j \otimes \boldsymbol{e}_B, \quad (7.4)$$

and where $U = (u_{ij}) \in U(2)$, $V = (v_{AB}) \in U(4)$. Hence, the matrix C defined in (7.3) transforms according to

$$C \mapsto UCV^T. \quad (7.5)$$

It is now easy to see that $\det(CC^*)$ is invariant under the $U(2)\times U(4)$ action. Hence, $\det(CC^*)$ may serve as (squared) concurrence (up to a constant factor) between one-qubit and the other two-qubit.

If Ψ is separable in the sense discussed above, there exist non-vanishing vectors $(c_j) \in \mathbf{C}^2$ and $(d_A) \in \mathbf{C}^4$ such that $c_{jA} = c_j d_A$. Then, we may further treat the quantity $\left|\det\begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11}\end{pmatrix}\right|$ as a concurrence. If this quantity vanishes further, the state Ψ is fully separable in the sense that Ψ is a tensor product of three one-qubit states.

We now put $\det(CC^*)$ in another form. Let $\boldsymbol{c}_i = \sum_A c_{iA}\boldsymbol{e}_A \in \mathbf{C}^4$. Then, the state Ψ is separable in the sense that Ψ is a tensor product of a one-qubit state and a two-qubit state, if and only if c_i, $i = 0, 1$, are linearly dependent. It is well known that the vectors c_i, $i = 0, 1$, are linearly dependent, if and only if

$$\|c_0 \wedge c_1\|^2 = \det \begin{pmatrix} \langle c_0, c_0 \rangle & \langle c_0, c_1 \rangle \\ \langle c_1, c_0 \rangle & \langle c_1, c_1 \rangle \end{pmatrix} = 0, \tag{7.6}$$

where the first equality of the above equation gives the definition of the squared norm of the two-vector $c_0 \wedge c_1 \in \bigwedge^2 \mathbf{C}^4$. Thus, $\|c_0 \wedge c_1\|$ can serve as a concurrence between one-qubit and the other two-qubit [9, 16]. (Comments on this will be given in the next section.) This quantity is expressed in terms of c_{iA} as

$$\|c_0 \wedge c_1\|^2 = \sum_{A<B} \left| \det \begin{pmatrix} c_{0A} & c_{1A} \\ c_{0B} & c_{1B} \end{pmatrix} \right|^2, \tag{7.7}$$

and further proves to be equal to $\det(CC^*)$,

$$\det(CC^*) = \det \begin{pmatrix} \langle c_0, c_0 \rangle & \langle c_1, c_0 \rangle \\ \langle c_0, c_1 \rangle & \langle c_1, c_1 \rangle \end{pmatrix} = \|c_0 \wedge c_1\|^2. \tag{7.8}$$

If a three-qubit state Ψ is put in the form

$$\Psi = \sum_{A,\ell} c_{A\ell} e_A \otimes e_\ell, \tag{7.9}$$

in place of (7.2), the same argument applies to the coefficient matrix $(c_{Aj}) \in \mathbf{C}^{4\times 2}$. One can group the first and the third factors to form another coefficient matrix, to which a similar argument applies.

For four-qubit systems, we can put a state

$$\Psi = \sum_{j,k,\ell,m} c_{jk\ell m} e_j \otimes e_k \otimes e_\ell \otimes e_m \tag{7.10}$$

in different forms,

$$\Psi = \sum_{j,K} c_{jK} e_j \otimes e_K, \tag{7.11}$$

$$\Psi = \sum_{A,B} c_{AB} e_A \otimes e_B, \tag{7.12}$$

where

$$e_K = e_k \otimes e_\ell \otimes e_m, \qquad K \in \{000, 001, \ldots, 111\}, \tag{7.13}$$

$$e_A = e_j \otimes e_k, \quad e_B = e_\ell \otimes e_m, \qquad A, B \in \{00, \ldots, 11\}. \tag{7.14}$$

We associate (7.11) and (7.12) with the coefficient matrices

$$F = (c_{jK}) \in \mathbf{C}^{2\times 8}, \qquad G = (c_{AB}) \in \mathbf{C}^{4\times 4}, \tag{7.15}$$

respectively. Since Ψ is normalized, the matrices F and G are subject to the constraints

$$\mathrm{tr}(FF^*) = 1, \qquad \mathrm{tr}(GG^*) = 1, \tag{7.16}$$

respectively.

In the case of (7.11), the state is separable in the sense that Ψ is a tensor product of a one-qubit state and the other three-qubit state if and only if F is of rank 1. Since FF^* is of rank 1 or 2, FF^* is of rank 1 if and only if $\det(FF^*) = 0$. As F and FF^* has the same rank, the quantity $\det(FF^*)$ serves as a measure of entanglement, which is invariant under the $U(2) \times U(8)$ action on the state space, as is verified in the same manner as that for the three-qubit case (7.5).

In the case of (7.12), the state is separable in the sense that Ψ is a tensor product of a two-qubit state and another two-qubit state, if and only if the coefficient matrix $G = (c_{AB})$ is of rank 1. Since G is subject to the constraint $\mathrm{tr}(GG^*) = 1$, the sum of eigenvalues of GG^* is equal to 1. Hence, GG^* is of rank 1, if and only if one of the eigenvalues of the positive semi-definite matrix GG^* is 1, for which a necessary and sufficient condition is that $\det(I_4 - GG^*) = 0$, where I_4 denotes the 4×4 unit matrix. Thus, we may take $\det(I_4 - GG^*)$ as a measure of entanglement, which are invariant under the $U(4) \times U(4)$ action on the state space. In summary, we may take

$$\det(FF^*) \qquad \text{and} \qquad \det(I_4 - GG^*) \tag{7.17}$$

as measures of entanglement between one-qubit and the other three-qubit, and between two-qubit and another two-qubit, respectively.

Measures of entanglement for five- and more-qubit systems will be able to be defined in the same manner. First, we form a coefficient matrix $H \in \mathbf{C}^{2^\ell \times 2^m}$, where $\ell + m = n$ for n-qubit systems. Then, we describe the condition for H to be of rank 1. If $2 < \ell \leqslant m$, the condition takes the form $\det(I - HH^*) = 0$, where I denotes the $2^\ell \times 2^\ell$ unit matrix. If $2 = \ell \leqslant m$, the condition is written as $\det(HH^*) = 0$. Thus, $\det(I - HH^*)$ or $\det(HH^*) = 0$ serve as measures of entanglement between ℓ-qubit and the other m-qubit, according to whether $2 < \ell \leqslant m$ or $2 = \ell \leqslant m$.

Measures of entanglement are studied in [15, 17] on the basis of bipartite partition $\mathbf{C}^{2^\ell} \otimes \mathbf{C}^{2^m}$. However, the measure of the form $\det(I - HH^*)$, which is easy to use, does not seem to have been mentioned.

8. Concluding remarks and comments

We have shown that for a two-qubit state C with concurrence $r = |2 \det C|$ is distant from the separable states by $\frac{1}{2} \sin^{-1} r$ with respect to the naturally defined Riemannian metric. After having realized this fact, we can point out that $\frac{1}{2} \sin^{-1} r$ happens to be equal to the Schmidt angle mentioned in [4]. The geometric property of three- and more-qubit concurrence is reserved in future study.

In what follows, we make comments on three- and more-qubit concurrence, and give examples of the measures (7.17). We take up $\det(CC^*)$ for a three-qubit. We denote equation (7.1) in the Dirac notation by $|\Psi\rangle = \sum c_{jk\ell} |ik\ell\rangle$. The reduced density matrix ρ_A, i.e., the partial trace of $|\Psi\rangle\langle\Psi|$ over qubits B and C, is expressed as

$$\rho_A = \mathrm{tr}_{BC} |\Psi\rangle\langle\Psi| = \sum_{\ell,m} \sum_{j,k} c_{\ell jk} \overline{c_{mjk}} |\ell\rangle\langle m|, \tag{8.1}$$

which corresponds to the 2×2 matrix $\overline{CC^*}$. In contrast with this, the reduced density matrix ρ_{BC}, i.e., the partial trace of $|\Psi\rangle\langle\Psi|$ over qubit A, is associated with the 4×4 matrix $\overline{C^*C}$. In [9], $2\sqrt{\det \rho_A}$ is defined to be the concurrence between qubit A and the pair BC, and denoted by $C_{A(BC)}$. Since $\det(CC^*)$ is real-valued, one has $\det(CC^*) = \det \rho_A$, which shows that $\det(CC^*)$ is consistent with the concurrence defined in [9] as measures of entanglement. As for the matrix C^*C, since $\mathrm{rank}(C^*C) \leqslant 2$, the quantity $\det(C^*C)$ vanishes identically, so that it cannot serve as a measure of entanglement. However, in [9], they deal with C^*C by using the quantity $\mathrm{tr}(C^*C\overline{C^*C})$, which was defined to be $\mathrm{tr}(\rho_{BC}\widetilde{\rho}_{BC})$ in their notation. In our notation, we have

$$\mathrm{tr}(C^*C\overline{C^*C}) = |(\boldsymbol{c}_0, \boldsymbol{c}_0)|^2 + 2|(\boldsymbol{c}_0, \boldsymbol{c}_1)|^2 + |(\boldsymbol{c}_1, \boldsymbol{c}_1)|^2, \tag{8.2}$$

where (,) denotes the scalar product defined by $(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^{4} a_i b_i$.

In [16], Lévay uses the 2×2 matrices

$$C_0 = \begin{pmatrix} c_{000} & c_{001} \\ c_{010} & c_{011} \end{pmatrix}, \qquad C_1 = \begin{pmatrix} c_{100} & c_{101} \\ c_{110} & c_{111} \end{pmatrix} \tag{8.3}$$

in place of the vectors $\boldsymbol{c}_i, i = 0, 1$, and introduce the Plücker coordinates to express the complex plane spanned by C_0 and C_1 in the complex linear space $\mathbf{C}^{2\times 2}$. The quantities which appear in the right-hand side of (7.7) and denoted by det(∗) with labels A, B serve as the Plücker coordinates for the complex plane spanned by $\boldsymbol{c}_0, \boldsymbol{c}_1$. The Plücker coordinate method is extended for multi-qubit states [17, 18].

In conclusion, we apply the measures (7.17) for four-qubit states. For the GHZ state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, we have

$$\det(FF^*) = \tfrac{1}{4}, \qquad \det(I_4 - GG^*) = \tfrac{1}{4}. \tag{8.4}$$

For the W state $|\Psi\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$, we have

$$\det(FF^*) = \tfrac{1}{8}, \qquad \det(I_4 - GG^*) = \tfrac{1}{4}. \tag{8.5}$$

For the sake of comparison, we touch on a separate state $|\Psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$. A calculation gives

$$\det(FF^*) = \tfrac{1}{4}, \qquad \det(I_4 - GG^*) = 0. \tag{8.6}$$

The second equation of the above is trivial by definition. As in the case of three-qubits, we may define the concurrence between qubit A and the triple BCD to be $2\sqrt{\det \rho_A}$, where $\rho_A = \mathrm{tr}_{BCD}|\Psi\rangle\langle\Psi|$. Since the reduced density matrix ρ_A corresponds to $\overline{FF^*}$, the first equation of (8.6) means that the concurrence between qubit A and the triple BCD is equal to 1, which coincides with the concurrence for the two-qubit state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

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References

[1] Bennet C H, DiVincenzo D P, Smolin J and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[2] Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
[3] Mosseri R and Dandoloff R 2001 *J. Phys. A: Math. Gen.* **34** 10243
[4] Bengtsson I and Brännlund J 2002 *Int. J. Mod. Phys.* **17** 4675
[5] Lévay P 2004 *J. Phys. A: Math. Gen.* **37** 1821
[6] Meyer D A and Wallach N R 2002 *J. Math. Phys.* **43** 4273
[7] Carteret H A and Sudbery A 2000 *J. Phys. A: Math. Gen.* **33** 4981
[8] Knapp A W 2002 *Lie Groups Beyond an Introduction* 2nd edn (Boston: Birkhäuser)
[9] Coffman V, Kundu J and Wootters W K 2000 *Phys. Rev.* A **61** 052306
[10] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[11] Wong A and Christensen N 2001 *Phys. Rev.* A **63** 044301
[12] Bernevig B A and Chen H D 2003 *J. Phys. A: Math. Gen.* **36** 8325
[13] Miyake A 2003 *Phys. Rev.* A **67** 012108
[14] Miyake A and Verstraete F 2004 *Phys. Rev.* A **69** 012101
[15] Emary C 2004 *J. Phys. A: Math. Gen.* **37** 8293
[16] Lévay P 2005 *Phys. Rev.* A **71** 012334
[17] Lévay P 2005 *J. Phys. A: Math. Gen.* **38** 9075
[18] Heydari H 2006 *J. Math. Phys.* **47** 012103